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# The Packing Chromatic Number of Random $d$ -regular Graphs

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THE PACKING CHROMATIC NUMBER OF RANDOM  $d$ -REGULAR GRAPHS

by

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Bachelor of Arts  
East Carolina University 2011

Master of Arts  
East Carolina University 2013

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## ABSTRACT

Let  $G = (V(G), E(G))$  be a simple graph of order  $n$  and let  $i$  be a positive integer.  $X_i \subseteq V(G)$  is called an  $i$ -packing if vertices in  $X_i$  are pairwise distance more than  $i$  apart. A *packing coloring* of  $G$  is a partition  $X = \{X_1, X_2, X_3, \dots, X_k\}$  of  $V(G)$  such that each color class  $X_i$  is an  $i$ -packing. The minimum order  $k$  of a packing coloring is called the *packing chromatic number* of  $G$ , denoted by  $\chi_\rho(G)$ . Let  $G_{n,d}$  denote the random  $d$ -regular graph on  $n$  vertices. In this thesis, we show that for any fixed  $d \geq 4$ , there exists a positive constant  $c_d$  such that

$$\mathbf{P}(\chi_\rho(G_{n,d}) \geq c_d n) = 1 - o_n(1).$$

Keywords: packing chromatic number, random  $d$ -regular graphs, configuration model

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# CHAPTER 1

## BASIC DEFINITIONS

A *graph*  $G$  is a finite nonempty set of objects, called *vertices* (singular vertex), together with a (possibly empty) set of unordered pairs of distinct vertices, called *edges*. The set of vertices of the graph  $G$  is called the *vertex set* of  $G$ , denoted by  $V(G)$ , and the set of edges is called the *edge set* of  $G$ , denoted by  $E(G)$ . The edge  $e = \{u, v\}$  is said to *join* the vertices  $u$  and  $v$ . If  $e = \{u, v\}$  is an edge of  $G$ , then  $u$  and  $v$  are *adjacent vertices*, while  $u$  and  $e$  are *incident*, as are  $v$  and  $e$ . Furthermore, if  $e_1$  and  $e_2$  are distinct edges of  $G$  incident with a common vertex, then  $e_1$  and  $e_2$  are *adjacent edges*. It is convenient to henceforth denote an edge by  $uv$  or  $vu$  rather than by  $\{u, v\}$ . The cardinality of the vertex set of a graph  $G$  is called the *order* of  $G$  and is denoted by  $n(G)$ , or more simply by  $n$  when the graph under consideration is clear, while the cardinality of its edge set is the *size* of  $G$ , denoted by  $m(G)$  or  $m$ . A  $(n, m)$ -graph has order  $n$  and size  $m$ . The graph of order  $n = 1$  is called the *trivial graph*. A *nontrivial graph* has at least two vertices.

A *subgraph* of a graph  $G$  is a graph all of whose vertices belong to  $V(G)$  and all of whose edges belong to  $E(G)$ . If  $H$  is a subgraph of  $G$ , then we write  $H \subseteq G$ . If a subgraph  $H$  of  $G$  contains all the vertices of  $G$ , then  $H$  is called a *spanning subgraph* of  $G$ .

Let  $v$  be a vertex of a graph  $G$ . The *degree* of  $v$  is the number of edges of  $G$  incident with  $v$ . The degree of  $v$  is denoted by  $deg_G v$ , or simply  $d_G(v)$ . The *minimum degree* of  $G$  is the minimum degree among the vertices of  $G$  and is denoted  $\delta(G)$ , while the *maximum degree* of  $G$  is the maximum degree among the vertices of  $G$  and is denoted

$\Delta(G)$ .

A vertex is called *odd* or *even* depending on whether its degree is odd or even. A vertex of degree 0 in a graph  $G$  is called an *isolated vertex* and a vertex of degree 1 is an *end-vertex* of  $G$ . Of particular importance for us will be *regular graphs*. We say that a graph is *regular* if all its vertices have the same degree. In particular, if the degree of each vertex is  $d$ , then the graph is *regular of degree  $d$*  or is  *$d$ -regular*.

We say two graphs,  $G$  and  $H$ , are *isomorphic* if there is a one-to-one mapping  $\phi$  from  $V(G)$  onto  $V(H)$  such that  $\phi$  preserves adjacency; that is,  $uv \in E(G)$  if and only if  $\phi(u)\phi(v) \in E(H)$ . If  $G$  and  $H$  are isomorphic, then we write  $G \cong H$ .

A graph  $G$  is *connected* if there exists a path in  $G$  between any two of its vertices, and is *disconnected* otherwise. Every disconnected graph can be partitioned into connected subgraphs, called *components*. A component of a graph  $G$  is a maximal connected subgraph. Two vertices  $u$  and  $v$  in a graph  $G$  are *connected* if  $u = v$ , or if  $u \neq v$  and there is a  $u - v$  path in  $G$ . The number of components of  $G$  is denoted  $k(G)$ ; of course,  $k(G) = 1$  if and only if  $G$  is connected.

Let  $u$  and  $v$  be two (not necessarily distinct) vertices of a graph  $G$ . A  $u$ - $v$  walk in  $G$  is a finite, alternating sequence of vertices and edges that begin with the vertex  $u$  and ends with the vertex  $v$  and in which each edge of the sequence joins the vertex that precedes it to the vertex that follows it in the sequence. The number of edges in the walk is called the *length* of the walk. If all the edges of a walk are different, then the walk is called a *trail*. If, in addition, all the vertices are different, then the trail is called a *path*. A  $u$ - $v$  walk is *closed* if  $u = v$  and *open* otherwise. A closed walk in which all the edges are different is a *closed trail*. A closed trail which contains at least three vertices is called a *circuit*. A circuit which does not repeat any vertices (except the first and last) is called a *cycle*. The *length* of a cycle (or circuit) is the number of edges in the cycle (or circuit). A cycle of length  $n$  is an  *$n$ -cycle*. A cycle is *even* if its length is even; other wise it is *odd*. The minimum length of a cycle contained in a



graph  $G$  is the *girth* of  $G$ , denoted  $g(G)$ . The maximum length of a cycle in  $G$  is the *circumference*. If  $G$  contains no cycle then we define  $g(G) = \infty$  and its circumference to be 0.

An *acyclic* graph, one that does not contain a cycle, is called a *forest*. If a forest is connected, we say it is a *tree*. The vertices of degree 1 in a tree are called its *leaves*.

For a connected graph  $G$ , we define the *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  as the minimum of the lengths of the  $u - v$  paths of  $G$ . If  $G$  is a disconnected graph, then the distance between two vertices in the same component of  $G$  is defined as above. However, if  $u$  and  $v$  belong to different components of  $G$ , then  $d(u, v)$  is undefined. The greatest distance between any two vertices in  $G$  is the *diameter* of  $G$ , denoted by  $\text{diam}(G)$ . It is easy to see that if  $G$  contains a cycle then  $g(G) \leq 2\text{diam}(G) + 1$ . We define  $G^i = (V, E^i)$  where  $E^i = \{(u, v) | d(u, v) \leq i \text{ in } G\}$ .

The *open neighborhood* of a vertex  $v$  is  $N(v) = \{u \in V | uv \in E\}$ . In general, we define the *open neighborhood of a subset*  $X \subseteq V$  by  $N(X) = \{u \in V \setminus X | \exists v \in X, uv \in E(G)\}$ . The *closed neighborhood* of a vertex  $v$  is  $N[v] = \{v\} \cup N(v)$  and in general, the *closed neighborhood of a subset*  $X \subseteq V$  by  $N[X] = X \cup N(X)$ .

A *finite probability space* is a finite set  $\Omega \neq \emptyset$  together with a function  $\mathbf{P} : \Omega \rightarrow \mathbb{R}_{\geq 0}$  such that (1) for all  $\omega \in \Omega$ ,  $\mathbf{P}(\omega) > 0$  and (2)  $\sum_{\omega \in \Omega} \mathbf{P}(\omega) = 1$ . The set  $\Omega$  is called the *sample space* and the function  $\mathbf{P}$  is the *probability distribution*. An *event*  $E$  is a subset of  $\Omega$ . For  $E \subseteq \Omega$ , we define the *probability* of  $E$  to be  $\mathbf{P}(E) = \sum_{\omega \in E} \mathbf{P}(\omega)$ . We note that  $\mathbf{P}(\emptyset) = 0$  and  $\mathbf{P}(\Omega) = 1$ .

## CHAPTER 2

### THE PACKING CHROMATIC NUMBER

Let  $G = (V(G), E(G))$  be a simple graph of order  $n$  and let  $i$  be a positive integer.  $X_i \subseteq V(G)$  is called an *i-packing* if vertices in  $X_i$  are pairwise distance more than  $i$  apart. A *packing coloring* of  $G$  is a partition  $X = \{X_1, X_2, X_3, \dots, X_k\}$  of  $V(G)$  such that each color class  $X_i$  is an *i-packing*. Hence, two vertices may be assigned the same color if the distance between them is greater than the color. The minimum order  $k$  of a packing coloring is called the *packing chromatic number* of  $G$ , denoted by  $\chi_\rho(G)$ . Note that every packing coloring is a proper coloring.

Packing colorings were inspired by a frequency assignment problem in broadcasting. The distance between broadcasting stations is directly related to the frequency they may receive, since two stations may only be assigned the same frequency if they are located far enough apart for their frequencies not to interfere with each other. This coloring was first introduced by Goddard, Harris, Hedetniemi, Hedetniemi, and Rall [11] where it was called broadcast coloring. Brešar, Klavžar and Rall [5] were the first to use the term packing coloring.

Goddard et al. [11] investigated, amongst others, the packing chromatic number of paths, trees, and the infinite square lattice,  $\mathbb{Z}^2$ . They found that for the square lattice,  $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$ . In fact, the packing chromatic number of the square lattice received quite some attention in recent years. Fiala et al. [9] improved the lower bound to 10, and in [7], it is improved further to 12. Soukal and Holub [12] used a computer to better the upper bound to 17. The packing chromatic number of lattices, trees, and Cartesian products in general is also considered in [5] and [10].

Determining the packing chromatic number is considered to be difficult. In fact, finding  $\chi_\rho$  for general graph is NP-complete [11], and deciding whether  $\chi_\rho(G) \leq 4$  is also NP-complete. In [8], Fiala and Golovach showed that the decision whether a tree allows a packing coloring with at most  $k$  classes is NP-complete.

Jacobs, Jonck and Joubert in [13] examined the packing chromatic number of the Cartesian product of  $C_4$  and  $C_q$ . They proved, using a theoretical approach, that  $9 \leq \chi_\rho(C_4 \square C_q) \leq 11$  for  $q = 4t$  with  $t \geq 3$ . Lower bounds for the packing chromatic number of 3-regular graphs in have also been studied recently.

Of particular interest to us is the following theorem of Sloper [15] which shows that the packing chromatic number of the infinite binary tree is 7. He defines an *eccentric coloring* of a graph  $G$  in the following way.

An *eccentric coloring* of a graph  $G = (V, E)$  is a function  $color : V \rightarrow \mathbb{N}$  such that

1. For all  $u, v \in V$ ,  $(color(u) = color(v)) \Rightarrow d(u, v) > color(u)$
2. For all  $v \in V$ ,  $color(v) \leq e(v)$  where  $e(v) = \max_{u \in V} \{d(v, u)\}$

Note that the first condition is the definition of a packing coloring. A *complete binary tree* is a tree where all vertices have degree 1, 2, or 3.

**Theorem 2.1.** *Any complete binary tree of height of three or more is eccentrically colorable with 7 colors or less.*

His proof relies on the following definition:

An *expandable eccentric coloring* of a complete binary tree  $T = (V, E)$  is a coloring such that

1. For all  $u, v \in V$ ,  $(color(u) = color(v)) \Rightarrow d(u, v) > color(u)$
2. For all  $v \in V$ ,  $color(v) \leq e(v)$  where  $e(v) = \max_{u \in V} \{d(v, u)\}$
3. The root (level 1) is colored 1

4. All vertices on odd levels are colored 1
5. Every vertex colored 1 has at least one child colored 2 or 3
6.  $color(v) = 6$  and  $color(u) = 7$  implies  $d(u, v) \geq 5$
7.  $color(p) \in \{4, 5, 6, 7\}$  implies  $p$ 's children each have children colored 2 and 3
8. For all  $u \in V$ ,  $color(u) \leq 7$

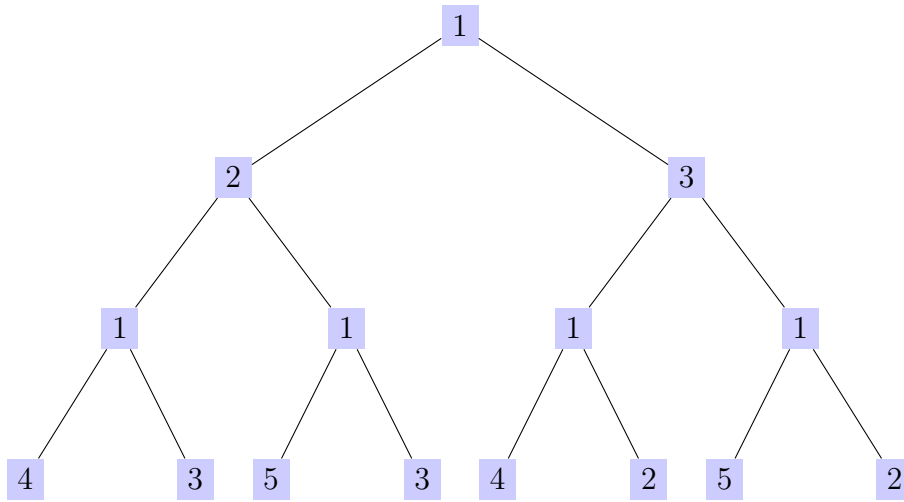


Figure 2.1 Expandable eccentric coloring

See Figure 2.1 for an example of an expandable eccentric coloring of a complete binary tree of height 4.

**Lemma 2.1.** *An expandable eccentric coloring of a complete binary tree of height  $n$  can be extended to an expandable eccentric coloring of height  $(n + 1)$ .*

*Proof.* We construct the eccentric coloring for the height  $(n + 1)$  tree by coloring the first  $n$  levels and showing that the vertices on the  $(n + 1)$  level can always be colored according to the expandable coloring rules.

First note, if  $n$  is even then by rule 4 of the definition, the vertices at level  $n - 1$  must all have color 1 and hence, the vertices at level  $n + 1$  may be colored 1.

So, we may assume  $n$  is odd. Note that all vertices at level  $n$  are colored 1 and hence no vertex at level  $n + 1$  may have the color 1. Consider a leaf  $u$  at level  $n + 1$  and its grandparent  $p$ . If  $color(p) \in \{4, 5, 6, 7\}$  then  $u$  and its sibling, say  $v$ , are assigned the colors 2 and 3 (order does not matter). See Figure 2.2.

We now consider the case when  $color(p) = 2$  (the case when  $color(p) = 3$  is handled similarly). Since for any grandchild  $j$  of  $p$ ,  $d(j, p) = 2$ , we have  $color(j) \neq 2$ . Let  $u, v, w, z$  be  $p$ 's grandchildren with pairs of siblings  $\{u, v\}$  and  $\{w, z\}$ . We consider all vertices at distance at most 6 from  $u, v, w$ , and  $z$  and on even levels. Note that any vertex at distance 7 from  $u, v, w$ , or  $z$  must be on an odd level and is already colored 1. By rule number 5, two of  $p$ 's grandchildren (which are not siblings) must receive the color 3. Without loss of generality suppose  $color(v) = color(z) = 3$ . For the rest of the proof, we refer to the labeling found in Figure 2.3.

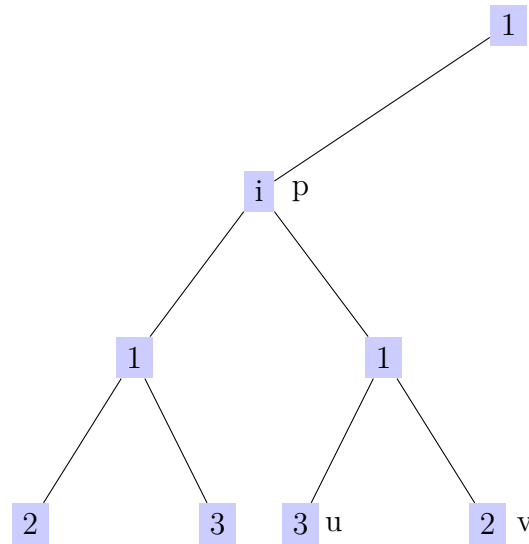


Figure 2.2 A coloring of level  $n + 1$  if  $color(p) \in \{4, 5, 6, 7\}$

Observe that the vertices  $g$  and  $h$  (and their siblings) are on level  $n + 1$ . If  $g$  and  $h$  have not been colored, they do not interfere with the coloring of  $u$  and  $w$ . Thus, we may assume that  $g$  and  $h$  have been colored according to the expandable coloring rules. Note that by rule 5,  $g, h$ , and  $c$ 's siblings must all be colored either 2 or 3.

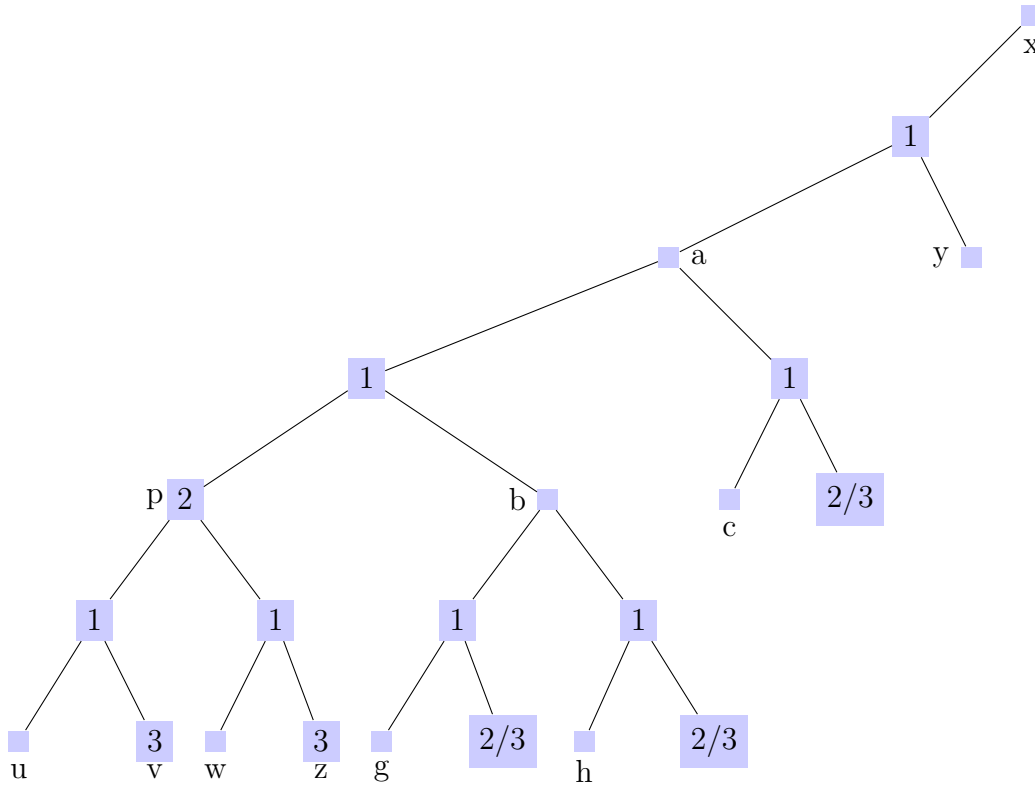


Figure 2.3 The subtree examined when  $color(p) = 2$

As  $color(u) \neq 2$  or  $3$  and similarly,  $color(w) \neq 2$  or  $3$ , by rule 8 we have  $color(u) \in \{4, 5, 6, 7\}$  and  $color(w) \in \{4, 5, 6, 7\}$ . We must show that we can always color  $u$  and  $w$  with these colors. There are four cases to consider depending on the color of  $p$ 's grandparent,  $a$ .

For convenience, we say that a vertex  $j$  *blocks* color  $\alpha$  from vertex  $k$  if and only if  $color(j) = \alpha$  and  $d(j, k) \leq \alpha$ , that is, coloring vertex  $k$  with color  $\alpha$  would violate rule 1.

**Case 2.1.**  $color(a) \in \{1, 2\}$

This is impossible due to rule 1.

**Case 2.2.**  $color(a) = 3$

As  $color(a) = 3$  and  $color(p) = 2$  we have that  $color(b) \notin \{2, 3\}$  and hence  $color(b) \in \{4, 5, 6, 7\}$ . Then, by rule 7,  $color(g) \in \{2, 3\}$  and  $color(h) \in \{2, 3\}$ . Note that  $d(u, x) = d(u, y) = d(u, c) = 6$  and similarly, the distance from  $w$  to  $x$ ,  $y$ , and  $c$  is also 6. Thus, the set  $\{x, y, c\}$  will block at most one color from  $\{u, w\}$ . As  $d(u, b) = d(w, b) = 4$ , and  $d(b, x) = d(b, y) = d(b, c) = 4$ , the vertex  $b$  will block a different color from  $\{u, w\}$ . This leaves at least two colors for  $u$  and  $w$ .

**Case 2.3.**  $color(a) \in \{4, 5\}$

Note, as  $color(a) \in \{4, 5\}$ , by rule 7 we have  $color(b) = 3$ ,  $color(c) \in \{2, 3\}$ , and  $color(x) \in \{2, 3\}$ . By rule 5, we have  $color(y)$  is also either 2 or 3. Thus, we have either  $color(g) \in \{4, 5\}$  and  $color(h) \in \{6, 7\}$  or vice versa but not both by rule 6. Hence, there are at least two colors for  $u$  and  $w$  contained in the set  $\{4, 5, 6, 7\}$  as  $d(u, \{g, h\}) = d(w, \{g, h\}) = 6$ .

**Case 2.4.**  $color(a) \in \{6, 7\}$

Since  $color(a) \in \{6, 7\}$ , by rule 7 we have  $color(b) = 3$ ,  $color(c) \in \{2, 3\}$ , and  $color(x) \in \{2, 3\}$ . By rule 5, we have  $color(y)$  is also either 2 or 3. As  $d(u, \{g, h\}) = d(w, \{g, h\}) = 6$ , the vertices  $g$  and  $h$  can not block colors 4 and 5 from  $u$  and  $w$ . Thus, we may color the vertices  $u$  and  $w$  with colors 4 and 5.

Therefore, we have shown it is possible to color any leaf at level  $n + 1$  according to the expandable coloring rules, assuming the other leaves are colored according to the rules or are uncolored. Hence, we may color all of level  $n + 1$ .  $\square$

The proof of Theorem 2.1 follows immediately by induction using the example in Figure 2.1 (without leaves) as a base case.

Surprisingly, Sloper's theorem can not be extended to *complete  $k$ -ary trees* with  $k \geq 3$ . A  $k$ -ary tree is a tree  $T$  such that for all  $v \in V(T)$ ,  $d_T(v) \leq k + 1$ . We can inductively define the complete  $k$ -ary tree,  $T_k$ :

1.  $T_1 := 1$  vertex, the root
2.  $T_i :=$  Start with  $T_{i-1}$  and append  $k$  new leaves to each leaf of  $T_{i-1}$ .

The *height* of a complete  $k$ -ary tree is  $h = d(\text{root}, \text{leaf}) + 1$ .

An *eccentric broadcast coloring* of a graph  $G = (V, E)$  is a function  $\text{color} : V \rightarrow \mathbb{N}$  such that

1. For all  $u, v \in V$ ,  $(\text{color}(u) = \text{color}(v)) \Rightarrow d(u, v) > \text{color}(u)$
2. For all  $v \in V$ ,  $\text{color}(v) \leq \text{diam}(G)$

**Theorem 2.2.** *No complete  $k$ -ary tree,  $k \geq 3$ , of height  $h$ ,  $h \geq 4$  is eccentrically broadcast-colorable.*



## CHAPTER 3

### RANDOM GRAPHS

Erdős and Rényi are given credit for first implementing the use of random graphs in probabilistic proofs of the existence of graphs with special properties such as arbitrarily large girth and chromatic number which had not been found constructively at the time. The study of random regular graphs grew in popularity much later with the works of Bender and Canfield, Bollobás, and Wormald. The study of random graphs has in part grown due to developments in computer science. Random graphs have applications in all areas in which complex networks need to be modeled.

Due to their relation to statistics, the first combinatorial structures to be studied probabilistically were tournaments. In 1943, Szele applied probabilistic methods to extremal problems in combinatorics. While it is not easy to construct a tournament of order  $n$  with many Hamilton paths, Szele was able to show the existence of a tournament of order  $n$  with at least  $n!/2^{n-1}$  Hamilton paths since this is the value of the expected number of Hamilton paths. Erdős used similar arguments to give a lower bound on the Ramsey number  $R(k)$ .

One of the most interesting discoveries of Erdős and Rényi was that many graph properties appear suddenly. That is, if we select a function  $F = F(n)$  then either almost every graph  $G_F$  has property  $P$  or almost every graph does not have property  $P$ . The transition from a property being unlikely to very likely is, a lot of the time, very swift. Consider a monotone (increasing) property  $P$ , i.e., a graph has property  $P$  whenever one of its subgraphs has property  $P$ . Then, for some properties, we can find a threshold function  $F_0(n)$ . If  $F(n)$  grows a bit faster or slower than  $F_0(n)$

then almost every  $G_F$  has or does not have property  $P$ , respectively. In [4], Bollobás provides as an example,  $F_0(n) = \frac{1}{2}n \log n$ , the threshold function for connectedness. If  $f(n) \rightarrow \infty$  then almost every  $G$  is disconnected for  $F(n) = \frac{1}{2}n(\log n - f(n))$  and almost every  $G$  is connected for  $F(n) = \frac{1}{2}n(\log n + f(n))$ .

In this paper, we will be concerned with random regular graphs. Random regular graphs in particular have applications in computer science and biogeography. Results on random regular graphs can often be extended to more general degree sequences. Perhaps the first result on short cycles in random regular graphs of degree at least 3 was given by Wormald in 1978. He determined the expected number of triangles in random cubic graphs using recurrence relations with the asymptotic result  $4/3$  obtained. This method of using recurrence relations and following with an asymptotic analysis has not been able to be extended much further for general problems. The method that has proven most fruitful has been a more direct probabilistic approach with an initially asymptotic viewpoint.

### 3.1 CONFIGURATION MODEL

Let  $\mathcal{G}_{n,d}$  denote the uniform probability space of  $d$ -regular graphs on the  $n$  vertices  $\{1, 2, \dots, n\}$  (where  $dn$  is even). Sampling from  $\mathcal{G}_{n,d}$  is equivalent to taking such a graph uniformly at random (u.a.r.).

We can define another uniform probability space,  $\mathcal{M}_{n,d}$ , of  $d$ -regular graphs on the  $n$  vertices  $\{1, 2, \dots, n\}$  (where  $dn$  is even) in the following way. Partition a set of  $dn$  points into  $n$  subsets  $v_1, v_2, \dots, v_n$  of  $d$  points each. Let  $M$  denote a perfect matching of the points into  $dn/2$  pairs. Then  $M$  corresponds to a multi-graph (with loops permitted)  $G_{n,d}(M)$  in which the subsets are now vertices and the pairs in the matching are edges, that is, a pair  $(u, v) \in M$  corresponds to an edge  $(x_i, y_j) \in G_{n,d}(M)$  where  $u \in x_i$  and  $v \in y_j$ . Note that each (simple) graph corresponds to  $(d!)^n$  matchings so a regular graph can be chosen u.a.r. by choosing a matching and

rejecting the result if it has loops or multiple edges. Note that non-simple graphs are not produced uniformly at random since for each loop the number of corresponding pairings is divided by 2, and for each  $k$ -tuple edge it is divided by  $k!$ . We may assume that the points are the elements of  $\{1, 2, \dots, n\} \times \{1, 2, \dots, d\}$  so that  $G_{n,d}(M)$  is induced by a projection. This model was first introduced by Bollobás and is given extensive treatment by Wormald in [16].

There are many ways to select a matching u.a.r. For instance, the points in the matching can be chosen sequentially. The first point in the next random pair chosen can be selected using any rule as long as the second point in that pair is chosen u.a.r. from the remaining points.

The configuration model for random regular graphs can be extended to random graphs with given degree sequence  $d_1, \dots, d_n$ . Let each subset  $v_i$  contain  $d_i$  points and select a perfect matching u.a.r. Restricting to no loops or multiple edges produces u.a.r. graphs with the desired degree sequence.

For any matching  $M$ , Bender and Canfield [1] show that the probability that  $G_{n,d}(M)$  is simple is given by

$$\mathbf{P}(G_{n,d}(M) \text{ simple}) = (1 + o(1)) \exp\left(\frac{1 - d^2}{4}\right) \text{ for fixed } d.$$

Now, since the number of perfect matchings on  $dn$  points is

$$\frac{(dn)!}{(dn/2)!2^{dn/2}},$$

the number of  $d$ -regular graphs on  $n$  vertices is

$$|\mathcal{G}_{n,d}| \sim \sqrt{2} \exp\left(\frac{1 - d^2}{4}\right) \left(\frac{d^d n^d}{e^d (d!)^2}\right)^{n/2}.$$

We note that  $f(n) \sim g(n)$  means  $f(n) = (1 + o(1))g(n)$  as  $n \rightarrow \infty$ . This result was found independently by Bender and Canfield [1] and Wormald [17]. In 1979, Bollobás gave a proof using the configuration model and showed that the formula applied for  $d = d(n) \leq \sqrt{2 \log n} - 1$ .

Using the estimation for  $\mathbf{P}(G_{n,d}(M) \text{ simple})$  and the relation between events in  $\mathcal{G}_{n,d}$  and matchings in  $\mathcal{M}_{n,d}$ , McKay and Wormald in [16] and [14] extend the previous results using a *switching* method. We do not discuss this method here but refer the reader to [16]. For an event  $H$  in  $\mathcal{M}_{n,d}$  define  $G(H)$  to be the event in  $\mathcal{G}_{n,d}$  containing all simple graphs of the form  $G(M)$  for some  $M \in H$ .

**Corollary 3.1.** *Let  $d \geq 1$  be fixed, and let  $H$  be an event which is a.a.s. true in  $\mathcal{M}_{n,d}$ . Then  $G(H)$  is a.a.s. true in  $\mathcal{G}_{n,d}$ .*

**Corollary 3.2.** *For  $d = o(\sqrt{n})$  the number of  $d$ -regular graphs on  $n$  vertices is*

$$\frac{(dn)!}{\left(\frac{1}{2}dn\right)! 2^{dn/2} (d!)^n} \exp\left(\frac{1-d^2}{4} - \frac{d^3}{12n} + O\left(\frac{d^2}{n}\right)\right).$$

The following theorem of Bollobás and de la Vega [3] will be used in the proof of our main result.

**Theorem 3.1.** *Let  $G$  be a  $d$ -regular random graph on  $n$  vertices. Then, with high probability, the diameter of  $G$  is*

$$\text{diam}(G) = D = (1 + o(1)) \log_{d-1}(n).$$

## CHAPTER 4

### MAIN RESULT

**Theorem 4.1.** *For any integer  $d \geq 4$ , there exists a positive constant  $c_d$  such that*

$$\mathbf{P}(\chi_\rho(G_{n,d}) \geq c_d n) = 1 - o_n(1).$$

We will need the following theorem and lemmas to prove our main result.

**Theorem 4.2.** *Let  $G$  be a  $d$ -regular graph with girth  $g$ . If  $d \geq 4$ , then  $\chi_\rho(G) \geq g - 1$ .*

*Proof.* Let  $k = g - 2$  and assume  $\chi_\rho(G) \leq k$ . Then there is a partition  $V = V_1 \cup V_2 \cup \dots \cup V_k$  such that for any  $1 \leq i \leq k$ , and two distinct vertices  $u, v \in V_i$ ,  $d(u, v) \geq i$ . For any vertex  $u$ , let  $N_i(u)$  be the set of vertices of distance at most  $i$  from  $u$ . Similarly, for any edge  $uv$ , let  $N_i(uv)$  be the set of vertices of distance at most  $i$  from  $u$  or  $v$ .

Note that the induced graph on  $N_i(u)$  (for  $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ ) is a tree depending only on  $i$ . Similarly, the induced graph of  $G$  on  $N_i(uv)$  (for  $i = 1, \dots, \lfloor \frac{k}{2} \rfloor - 1$ ) is a tree depending on  $i$ .

Thus,

$$|N_i(u)| = 1 + d + d(d-1) + \dots + d(d-1)^{i-1} = \frac{d(d-1)^i - 2}{d-2}$$

and

$$|N_i(uv)| = 2(1 + (d-1) + \dots + (d-1)^{i-1}) = \frac{2d(d-1)^i - 2}{d-2}.$$

Now, observe that  $|V_{2i} \cap N_i(u)| \leq 1$  for  $i = 1, 2, \dots, \lfloor \frac{k}{2} \rfloor$  and  $|V_{2i-1} \cap N_{i-1}(uv)| \leq 1$  for  $i = 1, 2, \dots, \lceil \frac{k}{2} \rceil$ . Also note that  $\cup_u N_i(u)$  and  $\cup_{uv} N_i(uv)$  cover all the vertices of  $G$  evenly.

Thus,

$$|V_{2i}| \leq \frac{n}{|N_i(u)|} = \frac{n(d-2)}{d(d-1)^i - 2} \quad \text{for } 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$$

and

$$|V_{2i-1}| \leq \frac{n}{|N_i(uv)|} = \frac{n(d-2)}{2d(d-1)^i - 2} \quad \text{for } 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor.$$

As the series  $\phi_i(d) := \sum_{i=1}^{\infty} \frac{d-2}{d(d-1)^i - 2}$  and  $\nu_i(d) := \sum_{i=1}^{\infty} \frac{d-2}{2d(d-1)^i - 2}$  converge and are decreasing functions of  $d$ , we have  $\phi_i(d) + \nu_i(d) < 1$  for all  $d \geq 4$ . Hence,

$$\begin{aligned} n &= \sum_{i=1}^k |V_i| \\ &= \sum_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} |V_{2i-1}| + \sum_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} |V_{2i}| \\ &\leq \sum_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{n(d-2)}{2d(d-1)^i - 2} + \sum_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{n(d-2)}{d(d-1)^i - 2} \\ &< n(\phi_i(d) + \nu_i(d)) \\ &< n, \end{aligned}$$

a contradiction. □

**Corollary 4.1.** *For any  $d \geq 4$  and any integer  $k$ , there exists a  $d$ -regular graph  $G$  with  $\chi_\rho(G) \geq k$ .*

In the configuration model, the induced graph of  $G$  on  $N_i(u)$  is a tree with high probability but we must account for the possibility of *overlaps*. We refer to the case in which two subsets, say  $v_i$  and  $v_j$ , contain vertices which are matched to vertices of a third subset,  $v_k$ , as an *overlap*. Note that two overlaps occur with probability less than  $\epsilon$ . Let  $D = \text{diam}(G_{n,d})$ . Define  $f_i(d) := \frac{d(d-1)^i - 2}{(d-2)}$  and  $g_i(d) := \frac{2d(d-1)^i - 2}{(d-2)}$ .

**Lemma 4.1.** *Let  $u$  be a vertex of  $G_{n,d}$  and let  $N_i(u)$  denote the set of vertices of distance at most  $i$  from  $u$  in  $G_{n,d}$  where  $1 \leq i \leq (1 - o(1))D/2$ . For fixed  $u$ , with probability  $1 - O(\frac{1}{n})$ ,  $|N_i(u)| = f_i(d)$ , with probability  $O(\frac{1}{n})$ ,  $|N_i(u)| = f_i(d) - 1$ , and with probability  $O(\frac{1}{n^2})$ ,  $|N_i(u)| \leq f_i(d) - 2$ .*

*Proof.* First observe that

$$\begin{aligned}
\mathbf{P}(|N_i(u)| = f_i(d)) &= \left(1 - \frac{d-1}{nd-1}\right) \left(1 - \frac{2d-3}{nd-3}\right) \cdots (1 - \xi) \\
&\geq 1 - \frac{d-1}{nd-1} - \frac{2d-3}{nd-3} - \frac{3d-5}{nd-5} - \cdots - \xi \\
&\geq 1 - \frac{c_i d}{nd} \\
&= 1 - \frac{c_i}{n}
\end{aligned}$$

for  $\xi := \frac{(2f_i(d) - 1)(d - 1) + 1}{nd - (2f_i(d) - 1)}$  and  $c_i = O(f_i^2(d))$ . Since  $i \leq (1 - o(1))D/2$ , we have  $f_i(d) = o(\sqrt{n})$  and  $c_i = o(n)$ . As

$$\mathbf{P}(\exists u \text{ s.t. } |N_i(u)| = f_i(d) - 2) \leq n \frac{c_i}{n^2} \leq \frac{c_i}{n} = o(1),$$

we need only consider the case in which there is one overlap. Hence,

$$\mathbf{P}(|N_i(u)| \leq f_i(d) - 1) \leq \left(\frac{c_i}{n}\right)^2.$$

□

The proof of Lemma 4.2 follows a similar argument.

**Lemma 4.2.** *Let  $uv$  be an edge of  $G_{n,d}$  and let  $N_i(uv)$  denote the set of vertices of distance at most  $i$  from  $u$  or  $v$  in  $G_{n,d}$ , where  $1 \leq i \leq (1 - o(1))D/2$ . With probability  $1 - o(1)$ , for all  $u, v$ ,  $|N_i(uv)| \in \{g_i(d), g_i(d) - 1\}$ .*

Next we consider the case when  $i \in [(1 - o(1))D/2, 3D/4]$ .

**Lemma 4.3.** *Let  $u$  be a vertex of  $G_{n,d}$  and let  $N_i(u)$  denote the set of vertices of distance at most  $i$  from  $u$  in  $G_{n,d}$  where  $(1 - o(1))D/2 \leq i \leq 3D/4$ . With probability  $1 - o(1)$ , for all  $u$ ,  $f_i(d) - 4 \leq |N_i(u)| \leq f_i(d)$ .*

*Proof.* Let  $A_{v_1, \dots, v_\alpha}$  be the event that an overlap occurs at the subsets  $v_1, \dots, v_\alpha$ .

Then,

$$\mathbf{P}(A_{v_1, \dots, v_\alpha}) \leq \left(\frac{d-1}{nd-1-2f_i(d)}\right)^\alpha.$$

Thus,

$$\begin{aligned}
\mathbf{P}(|N_i(u)| = f_i(d) - \alpha) &\leq \mathbf{P}(\cup A_{v_1, \dots, v_\alpha}) \\
&\leq \sum_{v_1, \dots, v_\alpha} \mathbf{P}(A_{v_1, \dots, v_\alpha}) \\
&\leq (f_i(d))^\alpha \left( \frac{d-1}{nd-1-2f_i(d)} \right)^\alpha \\
&\leq c \cdot n^{\frac{3}{4}\alpha} n^{-\alpha} \\
&\leq c \frac{1}{n^{\alpha/4}} \\
&= o\left(\frac{1}{n}\right),
\end{aligned}$$

since  $i \leq 3D/4$  we have  $f_i(d) \leq n^{\frac{3}{4}}$ . Thus, for all  $u$ ,

$$\mathbf{P}(\exists u \text{ such that } |N_i(u)| \leq f_i(d) - \alpha) \leq no(1/n) = o(1).$$

Hence, choosing  $\alpha = 5$ , we have with probability  $1 - o(1)$  for all  $u$ ,  $|N_i(u)| \geq f_i(d) - 4 \geq \frac{1}{2}f_i(d)$ .

□

Again, a similar argument proves the following Lemma.

**Lemma 4.4.** *Let  $uv$  be an edge of  $G_{n,d}$  and let  $N_i(uv)$  denote the set of vertices of distance at most  $i$  from  $u$  or  $v$  in  $G_{n,d}$ , where  $(1 - o(1))D/2 \leq i \leq 3D/4$ . With probability  $1 - o(1)$ , for all  $u, v$ ,  $g_i(d) - 4 \leq |N_i(uv)| \leq g_i(d)$ .*

We may now prove the main result.

*Proof.* By Theorem 4.2, we may choose  $\epsilon > 0$  small enough so that  $\sum_{i=1}^{\infty} \frac{1}{f_i(d)} + \sum_{i=1}^{\infty} \frac{1}{g_i(d)} \leq 1 - \epsilon$ . Now, there exists an  $M$  such that

$$\sum_{i=M}^{\infty} \frac{1}{f_i(d)} + \sum_{i=1}^{\infty} \frac{1}{g_i(d)} < \frac{\epsilon}{6}.$$

By Theorem 3.1, we may choose  $n_0$  such that for  $n \geq n_0$ ,  $D = (1 + o(1)) \log_{d-1}(n) \geq 4M$ .



Recall  $G^i = (V, E^i)$  where  $E^i = \{(u, v) : d(u, v) \leq i \text{ in } G\}$ .

Let  $i \in [3D/4 + 1, D]$ . Note that as  $\alpha(G_{n,d}^i) \leq \alpha(G_{n,d}^j)$  for  $i \geq j$ , where  $\alpha$  is the independence number, we have that  $|V_i| \leq \alpha(G_{n,d}^i) \leq \alpha(G_{n,d}^j) \leq \frac{2}{f_j(d)}n$  for  $i \geq j$ .

Thus,  $\sum_{i=3/4D}^D |V_i| \leq \sum_{j=D/2}^{3/4D} \frac{2}{f_j(d)}n < \frac{\epsilon}{6}n$ .

When  $i > D$  observe that  $|V_i| \leq 1$ .

Thus,

$$\begin{aligned}
n &= \sum_{i=1}^k |V_i| \\
&= \sum_{i=1}^{(1-o(1))D/2} |V_i| + \sum_{i=(1+o(1))D/2+1}^{3D/4} |V_i| + \sum_{i=3D/4+1}^D |V_i| + \sum_{i>D} |V_i| \\
&< (1 - o(1/n)) \sum_{i=1}^{(1+o(1))D/2} \left( \frac{1}{f_i(d)} + \frac{1}{g_i(d)} \right) \\
&\quad + o(1/n) \sum_{i=1}^{(1+o(1))D/2} \left( \frac{1}{f_i(d) - 1} + \frac{1}{g_i(d) - 1} \right) \\
&\quad + \sum_{i=(1-o(1))D/2}^{3D/4} \left( \frac{1}{f_i(d) - 5} + \frac{1}{g_i(d) - 5} \right) + \sum_{j=D/2}^{3/4D} \frac{2}{f_j(d)}n + \sum_{i>D} |V_i| \\
&< (1 - \epsilon)n + \frac{\epsilon}{6}n + \frac{\epsilon}{6}n + \frac{\epsilon}{6}n + \sum_{i>D} |V_i| \\
&\leq \left(1 - \frac{\epsilon}{2}\right)n + \sum_{i>D} |V_i|
\end{aligned}$$

Thus,  $\sum_{i>D} |V_i| \geq \frac{\epsilon}{2}n$  and hence, with high probability,  $\chi_\rho(G_{n,d}) \geq c_d n$  for some constant  $c_d$ .  $\square$

In the future, we would like to determine how the packing chromatic number of random cubic graphs behaves.

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